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High-temperature representation of anisotropic rotator, XY and Heisenberg models for dimensions $D \geqslant 2$†

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Abstract. We establish a relation between discrete excitation models at low temperature and the anisotropic rotator, XY and Heisenberg models at high temperature for dimensions $D \geqslant 2$. The correlation functions of these models are shown to decay exponentially for short-range interactions. A qualitative discussion of the phase transitions is given.

1. Introduction

In a recent Letter (Holz 1978a) a discrete excitation model was introduced with a finite number of degrees of freedom per lattice site which can be used for a study of the high-temperature properties of the planar rotator model in dimensions $D \geqslant 2$. In the following a generalisation of this discrete excitation model will be presented which also allows the calculation of the high-temperature properties of the Ising-like models listed in the title of the paper. The procedure is based on the extension of the discrete excitation model introduced by Knops (1977) for $D = 2$ to arbitrary dimensions and the notion of monopoles associated with anisotropy fields.

The concept of monopoles used here was first introduced into the problem by Villain (1975) and independently developed by José *et al* (1977), and has recently been generalised by Kadanoff (1978).

The present procedure makes use of an approximate duality between the low-temperature properties of the discrete excitation models and the high-temperature properties of the Ising-like models. The degree of accuracy of the mapping between the properties of the two classes of models improves the more extreme the temperatures are. It is therefore not possible to give a completely satisfactory discussion of the phase transition (PT) occurring in the Ising-like models in the context of the present method, although that would be highly desirable.

The paper is presented as follows. In § 2 the basic formalism is developed. The correlation functions of some anisotropic rotator models are given in § 3. The extension of the formalism to the XY and Heisenberg models is developed in §§ 4 and 5 respectively. In § 6 the phase transitions which can occur in the discrete excitation models are discussed qualitatively with a look at those occurring in the Ising-like models.

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2. Basic formalism

We consider a Hamiltonian of the form

$$H_{\mathrm{DE}} = \tfrac{1}{2} \sum_{\langle i,j,\ldots\rangle} J^{\mathrm{i}}_{\mathrm{DE}} n^2_{\langle i,j,\ldots\rangle} + \sum_{i'} J^{\mathrm{a}}_{\mathrm{DE}} n^2_{i'} \tag{1}$$

on a D-dimensional simple hypercubic lattice with N lattice sites. Here $i, j, \ldots$ label lattice sites on the simple D-dimensional hypercubic lattice, and i' labels lattice sites on the dual lattice. $J^{\mathrm{i}}_{\mathrm{DE}}$ and $J^{\mathrm{a}}_{\mathrm{DE}}$ represent the isotropic and anisotropic coupling constants respectively. The symbol $\langle i, j, \ldots\rangle$ contains 2^{D-1} indices and represents the $(D-1)$-dimensional hypercubic face formed by the lattice sites $i_1, i_2, \ldots, i_{2^{D-1}}$, where i_j, i_{j+1} are nearest neighbours and $i_{2^{D-1}+1} = i_1$. Each hypercubic face appears in equation (1) twice, once oriented with an even permutation $\langle P^{\mathrm{e}} i_1, i_2, \ldots, i_{2^{D-1}}\rangle$ and once oriented with an odd permutation $\langle P^{\mathrm{o}} i_1, i_2, \ldots, i_{2^{D-1}}\rangle$. In the following this will be referred to as positive and negative orientation respectively. Furthermore we require

$$n_{\langle P^{\mathrm{e}} i_1, i_2, \ldots, i_{2^{D-1}}\rangle} = -n_{\langle P^{\mathrm{o}} i_1, i_2, \ldots, i_{2^{D-1}}\rangle} \tag{2}$$

and

$$n_{\langle i_1, i_2, \ldots, i_{2^{D-1}}\rangle} = 0, \pm 1. \tag{3}$$

The second term of equation (1) is referred to the dual lattice, and here we require

$$n_{i'} = 0, \pm p. \tag{4}$$

Because each lattice site i' of the dual lattice is surrounded by $2D$ $(D-1)$-dimensional hypercubic faces we can formulate the conservation law

$$\sum_{\langle i,j,\ldots\rangle} n^{i'}_{\langle i,j,\ldots\rangle} + n_{i'} = 0 \tag{5}$$

for each dual lattice site i'. Here $n^{i'}_{\langle i,j,\ldots\rangle}$ is given by

$$n^{i'}_{\langle i,j,\ldots\rangle} = n_{\langle P^{\mathrm{e(o)}} i_1, i_2, \ldots, i_{2^{D-1}}\rangle} \tag{6}$$

for the $2D$ hypercubic faces surrounding the dual site i', otherwise it vanishes. Furthermore, for a given site i' the orientation of each surrounding face is fixed once for always, so that for neighbouring sites i', j' the orientation of the hypercubic face shared by them appears in equation (6) for i' and j' with opposite signs. The problem defined by equations (1)–(6) will be referred to in the following as the discrete excitation model.

Using the integral representation for equation (5),

$$(2\pi)^{-1} \int_0^{2\pi} \mathrm{d}\phi_{i'} \exp\left[\mathrm{i}\phi_{i'}\left(\sum_{\langle i,j,\ldots\rangle} n^{i'}_{\langle i,j,\ldots\rangle} + n_{i'}\right)\right] = \delta_{(\sum_{\langle i,j,\ldots\rangle} n_{\langle i,j,\ldots\rangle} + n_{i'}),0}, \tag{7}$$

where the right-hand side denotes the Kronecker symbol, the partition function for the problem can be written in the form

$$Z_{p,\mathrm{R}} = (2\pi)^{-N} \prod_{i'=1}^{N} \int_0^{2\pi} \mathrm{d}\phi_{i'} \exp\left[-\beta_{\mathrm{DE}}\left(\sum_{\langle i',j'\rangle} \tilde{V}^{\mathrm{i}}(\phi_{i'} - \phi_{j'}) + \sum_{i'} \tilde{V}^{\mathrm{a}}(\phi_{i'})\right)\right], \tag{8}$$

where $\langle i', j'\rangle$ indicates summation over nearest neighbours on the dual lattice,

$$\tilde{V}^{\mathrm{i}}(\phi) = -(2\beta_{\mathrm{DE}})^{-1} \ln[1 + 2\cos\phi \exp(-\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}})], \tag{9}$$

$$\tilde{V}^{\mathrm{a}}(\phi) = -\beta^{-1}_{\mathrm{DE}} \ln[1 + 2\cos p\phi \exp(-\beta_{\mathrm{DE}} P^2 J^{\mathrm{a}}_{\mathrm{DE}})] \tag{10}$$

are effective potentials, and β_{DE} indicates the inverse temperature of the discrete excitation model. For

$$\beta_{\mathrm{DE}} J_{\mathrm{DE}}^{\mathrm{i(a)}} \gg 1 \tag{11}$$

equations (9) and (10) can be Taylor-expanded and we obtain approximately

$$Z_{p,\mathrm{R}} \sim Z_p^{\mathrm{R}} = (2\pi)^{-N} \prod_{i'=1}^{N} \int_0^{2\pi} \mathrm{d}\phi_{i'} \exp(-\beta_{\mathrm{R}} H_{\mathrm{R}}(\{\phi_{i'}\})). \tag{12}$$

Here the following abbreviations have been introduced:

$$H_{\mathrm{R}}(\{\phi_{i'}\}) \equiv -J_{\mathrm{R}}^{\mathrm{i}} \sum_{\langle i',j'\rangle} \cos(\phi_{i'} - \phi_{j'}) - J_{\mathrm{R}}^{\mathrm{a}} \sum_{i'} \cos p\phi_{i'}, \tag{13a}$$

$$\beta_{\mathrm{R}} J_{\mathrm{R}}^{\mathrm{i}} \equiv \exp(-\beta_{\mathrm{DE}} J_{\mathrm{DE}}^{\mathrm{i}}), \tag{13b}$$

$$\beta_{\mathrm{R}} J_{\mathrm{R}}^{\mathrm{a}} \equiv 2 \exp(-\beta_{\mathrm{DE}} J_{\mathrm{DE}}^{\mathrm{a}}). \tag{13c}$$

It follows from equations (11)–(13c) that the low-temperature partition function $Z_{p,\mathrm{R}}$ of the discrete excitation model maps onto the high-temperature partition function Z_p^{R} of the anisotropic rotator model, equation (13*a*). Here $J_{\mathrm{R}}^{\mathrm{i}}, J_{\mathrm{R}}^{\mathrm{a}}$ represent the coupling constants of the isotropic and anisotropic parts of the rotator model, and β_{R} denotes its inverse temperature. Raising the second subscript of the partition function $Z_{p,\mathrm{X}}$ of a discrete excitation model is always used in the following to indicate the partition function Z_p^X of the corresponding Ising-like model.

Before we proceed we would like to point out that the present formalism can be generalised in two ways. Firstly, anisotropic models where the second term of equation (13*a*) is substituted by $\Sigma_{\alpha=1}^{p} - J_{\mathrm{R}}^{\mathrm{a}}(\alpha) \Sigma_{i'} \cos \alpha\phi_{i'}$ can be obtained as follows. We substitute the second term of equation (1) by $\Sigma_{\alpha=1}^{p} \Sigma_{i'} J_{\mathrm{DE}}^{\mathrm{a}}(\alpha) n_{i'}^2(\alpha)$ and equation (4) by p equations $n_{i'}(\alpha) = 0, \pm\alpha$. This leads to $\beta_{\mathrm{R}} J_{\mathrm{R}}^{\mathrm{a}}(\alpha) \equiv 2 \exp(-\beta_{\mathrm{DE}} J_{\mathrm{DE}}^{\mathrm{a}}(\alpha))$. Secondly, rotator models with long-range interaction can be obtained, where the first term of equation (13*a*) is substituted by $-\Sigma_{i',j'} J_{\mathrm{R}}^{\mathrm{i}}(|i'-j'|) \cos(\phi_{i'} - \phi_{j'})$ and where $|i'-j'|$ denotes the distance between the pair of sites i', j'. In order to achieve this we substitute the first term of equation (1) by $\frac{1}{2} \Sigma_{i',j'} J_{\mathrm{DE}}^{\mathrm{i}}(|i'-j'|) n_{i',j'}^2$, equation (2) by $n_{i',j'} = -n_{j',i'}$ and equation (5) by $\Sigma_{j' \neq i'} n_{i',j'} + n_{i'} = 0$. This leads to

$$\beta_{\mathrm{R}} J_{\mathrm{R}}^{\mathrm{i}}(|i'-j'|) = \exp(-\beta_{\mathrm{DE}} J_{\mathrm{DE}}^{\mathrm{i}}(|i'-j'|)). \tag{14}$$

It follows immediately that interaction constants $J_{\mathrm{DE}}^{\mathrm{i}}(|i'-j'|)$ which increase with distance $|i'-j'|$ have to be used in order to satisfy the condition $\beta_{\mathrm{DE}} J_{\mathrm{DE}}^{\mathrm{i}}(|i'-j'|) \gg 1$ on which equation (14) is based. This leads then to coupling constants $J_{\mathrm{R}}^{\mathrm{i}}(|i'-j'|)$ which decrease with distance $|i'-j'|$ as is usually required. Let us point out that the present formalism could have been used immediately at the beginning of the paper and would have made the introduction of a dual lattice redundant. However, in order to explain some different aspects of this problem in § 6, this more intricate notation has been introduced.

We consider next the partition fønction of the discrete excitation model. Equations (2) and (3) imply that each configuration can be specified by associating with each hypercubic face either no arrow or an arrow of unit length piercing it. The beginning and end of the arrow are fixed to the two neighbouring dual lattice sites which share the hypercubic face. Equations (4) and (5) then imply that on the dual lattice an equal number of sources and sinks is present where p arrows emanate and p arrows terminate respectively. Kirchhoff's law has to be satisfied everywhere. Accordingly all lines are oriented and either closed or connected to the sources and sinks. It follows now that $p \leqslant 2D$ has to hold, and no more than D oriented lines can cross at a dual lattice point.

The partition function for $p = 0$ is given by

$$Z_{0,\mathrm{R}} = \sum_{\{C_{\mathrm{L}}\}} \exp\left(-\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} \oint_{C_{\mathrm{L}}} \mathrm{d}s_{\mathrm{L}}\right), \tag{15}$$

where C_{L} is a collection of oriented and closed loops on the dual lattice of total length $c_{\mathrm{L}} = \oint \mathrm{d}s_{\mathrm{L}}$, and $\{C_{\mathrm{L}}\}$ denotes all possible loop collections. It should be observed that $\oint \mathrm{d}s_{\mathrm{L}}$ denotes here the contour integral not only over one loop but a collection of loops. For $J^{\mathrm{a}}_{\mathrm{DE}}(p) > 0$ we obtain

$$Z_{p,\mathrm{R}} = \sum_{\{C_{\mathrm{L}}\},\{[C_{\mathrm{S}}]_p\}} \exp\left[-\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} \oint_{C_{\mathrm{L}}} \mathrm{d}s_{\mathrm{L}} - \beta_{\mathrm{DE}}\left(J^{\mathrm{a}}_{\mathrm{DE}}\, 2p^2 n([C_{\mathrm{S}}]_p) + J^{\mathrm{i}}_{\mathrm{DE}} \sum_{i=1}^{p} \int_{C^{i}_{\mathrm{S}}} \mathrm{d}s^{i}_{\mathrm{S}}\right)\right]. \tag{16}$$

Here $[C_{\mathrm{S}}]_p = (C^1_{\mathrm{S}}, C^2_{\mathrm{S}}, \ldots, C^p_{\mathrm{S}})$ is a collection of p-fold lines, where each constituent of one p-fold line has the same orientation with respect to their common initial or end point. $n([C_{\mathrm{S}}]_p)$ gives the number of p-fold lines or the number of monopole pairs of strength p. $\{[C_{\mathrm{S}}]_p\}$ denotes all possible collections of p-fold lines.

Let us point out that for long-range interaction models we can proceed in a similar way to the above, the only difference being that now the arrows will connect arbitrary pairs of sites on the dual lattice. Because $J^{\mathrm{i}}_{\mathrm{DE}}(|i'-j'|)$ increases with distance $|i'-j'|$, a well-defined problem still exists.

3. Correlation function of rotator models

Consider now the problem that at sites $0'$ and r' there is a source and a sink respectively of strength q, i.e. that

$$\sum_{\langle i,j,\ldots\rangle} n^{i'}_{\langle i,j,\ldots\rangle} + n_{i'} = q\delta_{i',0'} - q\delta_{i',r'} \tag{5'}$$

for $i' = 1, \ldots, N$ holds instead of equation (5). Under the condition of equation (11) the partition function of the problem is then

$$Z_{p,\mathrm{R}}(0', r') \sim Z^{\mathrm{R}}_p(0', r')$$
$$= (2\pi)^{-N} \prod_{i'=1}^{N} \int_0^{2\pi} \mathrm{d}\phi_{i'} \exp[\mathrm{i}q(\phi_{0'} - \phi_{r'})] \exp(-\beta_{\mathrm{R}} H_{\mathrm{R}}(\{\phi_{i'}\})). \tag{17}$$

Accordingly we obtain

$$\langle \exp[\mathrm{i}q(\phi_{0'} - \phi_{r'})]\rangle_p = \langle \cos q(\phi_{0'} - \phi_{r'})\rangle_p \sim Z_{p,\mathrm{R}}(0', r')/Z_{p,\mathrm{R}}, \tag{18}$$

where the subscript p reminds us that a p-anisotropy is present. It follows from equation (5′) that $q \leqslant p + 2D$ is necessary in order that $Z_{p,\mathrm{R}}(0', r')$ should not vanish. In the following $q = 1$ will be considered mainly. Higher-order correlation functions can be constructed in the same manner by adding further source and sink terms to the right-hand side of equation (5′).

We consider next a number of special cases.

(*a*) Isotropic rotator model, $p=0$.

$$\langle\exp[\mathrm{i}(\phi_{0'}-\phi_{r'})]\rangle_0\sim\sum_{\{C_L\},\{C_S(0',r')\}}\exp\left[-\beta_{DE}J^{i}_{DE}\left(\oint_{C_L}\mathrm{d}s_L+\int_{0'}^{r'}\mathrm{d}s\right)\right]\Big/Z_{0,R}, \tag{19}$$

where $C_S(0', r')$ denotes an oriented line conformation with ends attached to $0'$ and r' and total length $c_S(0', r')=\int_{0'}^{r'}\mathrm{d}s$, and $\{C_S(0', r')\}$ denotes all possible line conformations, each, however, with the same orientation. The coupling constant of the rotator model is given by equation (13*b*). It follows from equation (15) that in the limit $\beta_{DE}J^{i}_{DE}\gg1$ the saddle point of $Z_{0,R}$ consists of a dilute gas of small loops. Similarly the saddle point of $Z_{0,R}(0', r')$ consists of a dilute gas of small loops and an essentially straight line connecting $0'$ and r'. Accordingly we obtain for $|0'-r'|\gg1$

$$\langle\cos(\phi_{0'}-\phi_{r'})\rangle_0\sim\exp(-\kappa|0'-r'|), \tag{20}$$

where $\kappa\sim\beta_{DE}J^{i}_{DE}=-\ln(\beta_R J^{i}_R)$ is the inverse correlation length.

(*b*) Rotator model in the presence of a magnetic field, $p=1$.

$$\begin{aligned}\langle\exp[\mathrm{i}(\phi_{o'}-\phi_{r'})]\rangle_1&\\ \sim\Bigg\{&\sum_{\substack{\{C_L\},\{C_S(0',r')\},\\ \{[C_S]_1\}}}\exp\left[-\beta_{DE}J^{i}_{DE}\left(\oint_{C_L}\mathrm{d}s_L+\int_{0'}^{r'}\mathrm{d}s\right)\right]\\ &\times\exp\left[-\beta_{DE}\left(J^{a}_{DE}\,2n([C_S]_1)+J^{i}_{DE}\int_{C^1_S}\mathrm{d}s_S\right)\right]\\ &+\sum_{\substack{\{C_L\},\{C_S(0'),C_S(r')\},\\ \{[C_S]_1\}}}\exp\left[-\beta_{DE}J^{i}_{DE}\left(\oint_{C_L}\mathrm{d}s_L+\int_{C_S(0')}\mathrm{d}s_S+\int_{C_S(r')}\mathrm{d}s_S\right)\right]\\ &\times\exp\left[-\beta_{DE}\left(J^{a}_{DE}\,2n([C_S]_1+1)+J^{i}_{DE}\int_{C^1_S}\mathrm{d}s_S\right)\right]\Bigg\}\Big/Z_{1,R}.\end{aligned} \tag{21}$$

The presence of the magnetic field produces a gas of monopoles of unit strength which can now be connected to the source and sink at $0'$ and r'. This generates the line conformations $C_S(0')$ and $C_S(r')$, where $C_S(0')$ starts at $0'$ and ends at an arbitrary point, and where $C_S(r')$ starts at an arbitrary point and ends at r'.

In the limit $\beta_{DE}J^{i}_{DE}\gg1$ and $|0'-r'|\gg1$ the first term of equation (21) shows exponential decay and therefore vanishes for $|0'-r'|\to\infty$. The second term of equation (21), however, gives a finite contribution. This can be seen as follows. To each configuration which appears in $Z_{1,R}$ we add a source term at $0'$ and an additional monopole of negative unit charge at an arbitrary point, and connect them by the oriented line $C_S(0')$. The same operation is performed simultaneously for the sink term. If the line conformations $C_S(0')$ and $C_S(r')$ are now constructed by means of a random walk process, we obtain the following estimate:

$$\lim_{|0'-r'|\to\infty}\langle\exp[\mathrm{i}(\phi_{0'}-\phi_{r'})]\rangle_1\sim\exp(-2\beta_{DE}J^{a}_{DE})/[\beta_{DE}J^{i}_{DE}-\ln(2D-1)]^2. \tag{22}$$

Physically this result means that the gas of monopoles generated by the magnetic field screens the long-range interaction in the system, because it short-cuts the field lines between test charges, i.e. the source and sink of strength $q=1$.

(*c*) Uniaxial anisotropy, $p = 2$.
In this case the source and sink at the sites $0'$ and r' respectively cannot be screened by the monopoles of strength $p = \pm 2$ as was the case for $p = \pm 1$. In fact, because the line conformations $\{[C_S]_2\}$ are two-fold and $q = 1$, it follows immediately from equation (5′) that all terms of $\langle \exp[i(\phi_{0'} - \phi_{r'})]\rangle_2$ contain one oriented line from $0'$ to r'. This implies exponential decay of the correlations. Similarly the correlation function $\langle \exp[2i(\phi_{0'} - \phi_{r'})]\rangle_2$ will have a term analogous to the second term of $\langle \exp[i(\phi_{0'} - \phi_{r'})]\rangle_1$ and therefore will assume a finite value for $|0' - r'| \to \infty$. In this case the monopole gas is able to screen the long-range interaction between the test charges of strength $q = \pm 2$. Finally, we point out that, if in addition to a p-anisotropy a magnetic field is present with an orientation along one of the minima of the anisotropy field, then long-range interactions are screened for test charges with $q = \pm 1$.

In principle the case of long-range interactions can be treated in a completely analogous fashion. In this case we define

$$\beta_{DE} J^{i}_{DE}(i', j') = \beta_{DE} J^{i}_{DE} + \alpha \ln |i' - j'|, \qquad i' \neq j', \tag{23}$$

and use of equation (14) leads to

$$\beta_R J^{i}_R(|i' - j'|) = [\exp(-\beta_{DE} J^{i}_{DE})]/r^{\alpha}, \tag{24}$$

where $\alpha > 0$ has to be used. For $\beta_{DE} J^{i}_{DE} \gg 1$ and $|0' - r'| \gg 1$ we obtain the estimate

$$\langle \cos(\phi_{0'} - \phi_{r'})\rangle_0 \sim [\exp(-\beta_{DE} J^{i}_{DE})]/r^{\alpha}. \tag{25}$$

Accordingly, a power-law decay is the most one can obtain for rotator models with long-range interaction.

Let us point out that equation (23) implies that the discrete excitation model used is actually a formal device because the coupling constants depend on the temperature. The same applies to the discrete excitation models introduced in the following sections.

4. Correlation functions of the XY model

In order to obtain the XY model, the coupling constants of the discrete excitation model are substituted as follows:

$$\beta_{DE} J^{i}_{DE}(i', j') = \beta_{DE} J^{i}_{DE} - \ln(\sin\theta_{i'} \sin\theta_{j'}), \tag{26a}$$

$$\beta_{DE} J^{a}_{DE}(i') = \beta_{DE} J^{a}_{DE} - \ln(f(\sin\theta_{i'})), \tag{26b}$$

where i', j' refer to neighbouring lattice sites on the dual lattice which are separated by their common hypercubic face $\langle i, j, \ldots\rangle$. The function $f(x)$ should satisfy $0 \leq f(x) < 1$. In addition, the measure of the partition function of the discrete excitation model is supplemented by the integral operator

$$\prod_{i'=1}^{N} \tfrac{1}{2} \int_0^{\pi} d\theta_{i'} \sin\theta_{i'}.$$

Using the same routine as before we obtain instead of equation (12)

$$Z_{p,XY} \sim Z_p^{XY} = (4\pi)^{-N} \prod_{i'=1}^{N} \int_0^{\pi} d\theta_{i'} \sin\theta_{i'} \int_0^{2\pi} d\phi_{i'} \exp(-\beta_{XY} H_{XY}(\{\phi_{i'}\}, \{\theta_{i'}\})). \tag{27}$$

Here the following abbreviations have been introduced:

$$H_{XY}(\{\phi_{i'}\}, \{\theta_{i'}\})$$

$$\equiv -J^{i}_{XY} \sum_{\langle i',j'\rangle} \sin\theta_{i'} \sin\theta_{j'} \cos(\phi_{i'}-\phi_{j'}) - J^{a}_{XY} \sum_{i'} f(\sin\theta_{i'}) \cos p\phi_{i'}, \tag{28a}$$

$$\beta_{XY} J^{i}_{XY} \equiv \exp(-\beta_{DE} J^{i}_{DE}), \tag{28b}$$

$$\beta_{XY} J^{a}_{XY} \equiv 2\exp(-\beta_{DE} J^{a}_{DE}). \tag{28c}$$

For the sake of simplicity we consider in the following only the case $f(\sin\theta_{i'}) \equiv 0$. The partition function of the discrete excitation model corresponding to equation (27) can now be written in the form

$$Z_{0,XY} = \sum_{\{C_L\}} \prod_{i'=1}^{N} \tfrac{1}{2} \int_0^{\pi} d\theta_{i'} \sin\theta_{i'} \prod_{i' \in C_L} \sin^2\theta_{i'} \exp\left(-\beta_{DE} J^{i}_{DE} \oint_{C_L} ds_L\right), \tag{29a}$$

and this leads approximately to

$$Z_{0,XY} \sim \sum_{\{C_L\}} (\tfrac{2}{3})^{C_L} \exp(-\beta_{DE} J^{i}_{DE} \oint_{C_L} ds_L). \tag{29b}$$

In equation (29*a*) $i' \in C_L$ picks up all points of the contour C_L only once. Accordingly the intersection points of the lines are counted with their proper multiplicity. Equation (29*b*) applies to a dilute gas of loops where only a small number of crossings of lines occurs. Using the notation

$$S_{i'} = (S^x_{i'}, S^y_{i'}, S^z_{i'}) \equiv (\cos\phi_{i'} \sin\theta_{i'}, \sin\phi_{i'} \sin\theta_{i'}, \cos\theta_{i'})$$

the correlation functions of the XY model can be constructed in the same fashion as earlier using equation (5′). We obtain

$$\langle S^x_{0'} S^x_{r'} + S^y_{0'} S^y_{r'} \rangle$$

$$\sim \sum_{\{C_L\},\{C_S(0',r')\}} \prod_{i'=1}^{N} \tfrac{1}{2} \int_0^{\pi} d\theta_{i'} \sin\theta_{i'} \prod_{i' \in C_L} \sin^2\theta_{i'} \prod_{i' \in C_S(0',r')} \sin^2\theta_{i'}$$

$$\times \exp\left[-\beta_{DE} J^{i}_{DE} \left(\int_{C_L} ds_L + \int_{0'}^{r'} ds\right)\right] \Big/ Z_{0,XY}. \tag{30}$$

Here $i' \in C_L$ and $i' \in C_S(0', r')$ means that i' picks up all points of the corresponding contours only once, implying that the intersection points of the contours C_L and $C_S(0', r')$ are counted with their proper multiplicity. For $\beta_{DE} J^{i}_{DE} \gg 1$ we obtain approximately from equation (30)

$$\langle S_{0'} \cdot S_{r'} \rangle \sim \sum_{\{C_L\},\{C_S(0',r')\}} (\tfrac{2}{3})^{C_L + C_S(0',r')+1} \exp\left[-\beta_{DE} J^{i}_{DE} \left(\oint_{C_L} ds_L + \int_{0'}^{r'} ds\right)\right] \Big/ Z_{0,XY}, \tag{31}$$

where use has been made of $\langle s^z_{0'} s^z_{r'} \rangle \equiv 0$. Furthermore, equation (31), like equation (29*b*), only applies to the case of a dilute gas of loops where only a small number of crossings of lines occurs. If such crossings are taken into account properly, then a term for the number of crossings $n(c)$ (in which c lines interact and where $1 < c \leq D$ holds) has to be introduced. A formula similar to equation (31) can then be derived which is more intricate, however, and in which some of the '$\frac{2}{3}$' factors are substituted by others which are also less than one in magnitude.

It follows from equation (31) that for $|0'-r'|\gg 1$ an exponential decay of correlations is obtained,

$$\langle \boldsymbol{S}_{0'} \cdot \boldsymbol{S}_{r'}\rangle \sim (\tfrac{2}{3})^{|0'-r'|} \exp(-\kappa|0'-r'|), \tag{32}$$

where $\kappa \sim \beta_{\rm DE} J^{\rm i}_{\rm DE} = -\ln \beta_{\rm XY} J^{\rm i}_{\rm XY}$ is the inverse correlation length. The study of the anisotropic XY models follows the same line of thought as demonstrated in § 3 for the anisotropic rotator models, and leads qualitatively to the same results. This problem will therefore not be pursued further.

5. The Heisenberg model

For the sake of simplicity we study in this section only the isotropic Heisenberg model. Using equation (1) without the second term but supplemented by the term

$$-J^{\rm z}_{\rm DE} \sum_{\langle i',j'\rangle} \cos\theta_{i'} \cos\theta_{j'},$$

and then proceeding as in § 4 for the XY model, we obtain for the partition function of this discrete excitation model

$$Z_{0,\rm H} \sim Z^{\rm H}_0 = (4\pi)^{-N} \prod_{i'=1}^{N} \int_0^{\pi} \mathrm{d}\theta_{i'} \sin\theta_{i'} \int_0^{2\pi} \mathrm{d}\phi_{i'} \exp(-\beta_{\rm H} H_{\rm H}(\{\phi_{i'}\}, \{\theta_{i'}\})), \tag{33}$$

where

$$\beta_{\rm H} H_{\rm H}(\{\phi_{i'}\}, \{\theta_{i'}\})$$

$$\equiv -\exp(-\beta_{\rm DE} J^{\rm i}_{\rm DE}) \sum_{\langle i',j'\rangle} \sin\theta_{i'} \sin\theta_{j'} \cos(\phi_{i'}-\phi_{j'})$$

$$-\beta_{\rm DE} J^{\rm z}_{\rm DE} \sum_{\langle i',j'\rangle} \cos\theta_{i'} \cos\theta_{j'}. \tag{34a}$$

For

$$\exp(-\beta_{\rm DE} J^{\rm i}_{\rm DE}) = \beta_{\rm DE} J^{\rm z}_{\rm DE} \tag{34b}$$

the isotropic Heisenberg model is obtained. The partition function now reads

$$Z_{0,\rm H} = \sum_{\{C_{\rm L}\}} \prod_{i'=1}^{N} \tfrac{1}{2} \int_0^{\pi} \mathrm{d}\theta_{i'} \sin\theta_{i'} \prod_{i'\in C_{\rm L}} \sin^2\theta_{i'}$$

$$\times \exp\left(-\beta_{\rm DE} J^{\rm i}_{\rm DE} \oint_{C_{\rm L}} \mathrm{d}s_{\rm L} + \beta_{\rm DE} J^{\rm z}_{\rm DE} \sum_{\langle i',j'\rangle} \cos\theta_{i'} \cos\theta_{j'}\right). \tag{35}$$

We introduce next a special correlation function of the continuous Ising model (Griffiths 1969) defined as

$$E(C_{\rm L}, C_{\rm S}(0', r')) \equiv \prod_{i'=1}^{N} \tfrac{1}{2} \int_0^{\pi} \mathrm{d}\theta_{i'} \sin\theta_{i'} \prod_{i'\in C_{\rm L}} \sin^2\theta_{i'}$$

$$\times \prod_{i'\in C_{\rm S}(0',r')} \sin^2\theta_{i'} \exp\left(\beta_{\rm DE} J^{\rm z}_{\rm DE} \sum_{\langle i',j'\rangle} \cos\theta_{i'} \cos\theta_{j'}\right) \Big/ Z_{\rm CI}, \tag{36}$$

where

$$Z_{\mathrm{CI}} \equiv \prod_{i'=1}^{N} \tfrac{1}{2} \int_0^{\pi} \mathrm{d}\theta_{i'} \sin\theta_{i'} \exp\Big(\beta_{\mathrm{DE}} J^{z}_{\mathrm{DE}} \sum_{\langle i',j' \rangle} \cos\theta_{i'} \cos\theta_{j'}\Big) \tag{37}$$

is the partition function of the continuous Ising model. For the isotropic Heisenberg model we have in the disordered state $\langle S^x_{0'} S^x_{r'} \rangle = \langle S^y_{0'} S^y_{r'} \rangle = \langle S^z_{0'} S^z_{r'} \rangle$; it is therefore possible to construct the correlation function in the same fashion as for the XY model. We obtain then straightforwardly

$$\langle \boldsymbol{S}_{0'} \cdot \boldsymbol{S}_{r'} \rangle \sim \frac{3}{2} \sum_{\{C_{\mathrm{L}}\},\{C_{\mathrm{S}}(0',r')\}} \exp\Big[-\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} \Big(\oint_{C_{\mathrm{L}}} \mathrm{d}s_{\mathrm{L}} + \int_{0'}^{r'} \mathrm{d}s\Big)\Big]$$
$$\times E(C_{\mathrm{L}}, C_{\mathrm{S}}(0', r')) \Big/ \sum_{\{C_{\mathrm{L}}\}} \exp\Big(-\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} \oint_{C_{\mathrm{L}}} \mathrm{d}s_{\mathrm{L}}\Big) E(C_{\mathrm{L}}, 0). \tag{38}$$

It follows from equations (36) and (37) that

$$0 \leqslant E(C_{\mathrm{L}}, C_{\mathrm{S}}(0', r')) \leqslant 1 \tag{39}$$

holds. Accordingly for $\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} \gg 1$ exponential decay of the correlations for $|0' - r'| \gg 1$ is again obtained.

Comparison of equations (31) and (38) reveals that the correlation function of the Heisenberg model has a more complicated structure than that of the XY model. Because the continuous Ising model which has been introduced has a ferromagnetic ground state, ferromagnetic fluctuations which appear as $\beta_{\mathrm{DE}} J^{z}_{\mathrm{DE}}$ increases will prevent the growth of large loops. This follows from equation (36) where the factors $\sin^2 \theta_{i'}$ along the loops assume small values for $\theta_{i'} \sim 0, \pi$. This problem will be considered from a different point of view in § 6.

6. Phase transitions and conclusions

We consider in this section certain aspects of the phase transitions (PT) which occur in the discrete excitation models and their consequences for the rotator models. Although it is not known whether the discrete excitation model undergoes a PT, one expects on physical grounds that for $J^{\mathrm{a}}_{\mathrm{DE}} = 0$ such a PT occurs. Presumably it is connected with the infinite extension of loops, and a random walk argument would lead to the estimate

$$\beta^{\mathrm{c}}_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} \leqslant \ln(2D - 1) \tag{40}$$

for the transition temperature. (The exponent $\frac{1}{2}$ in equation (23) of Holz (1978a) is wrong.) The partition function in equation (8) can now be written in the form

$$Z_{0,\mathrm{R}} = (2\pi)^{-N} \prod_{i'=1}^{N} \int_0^{2\pi} \mathrm{d}\phi_i \exp \prod_{\langle n',m' \rangle} [1 + 2\exp(-\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}}) \cos(\phi_{n'} - \phi_{m'})], \tag{41}$$

from which it follows that for

$$\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} > \ln 2 \tag{42}$$

the saddle points of $Z_{0,\mathrm{R}}$ can be classified into sectors in the same way as for the ferromagnetic ($J_{\mathrm{R}} > 0$) rotator model (Holz 1978a). For $\beta_{\mathrm{DE}} J_{\mathrm{DE}} < \ln 2$ this is no longer the case because equation (41) will also exhibit saddle points showing all sorts of antiferromagnetic orderings. At $\beta_{\mathrm{DE}} J^{\mathrm{i}}_{\mathrm{DE}} = \ln 2$ the excitation energy of these saddle points is infinitely large. One may hope therefore that if the presumed PT in the discrete excitation model occurs for $\beta^{c}_{DE}J^{i}_{DE} > \ln 2$ it may have some features in common with the PT occurring in the rotator model. An improved situation arises, of course, for $\beta^{c}_{DE}J^{i}_{DE} \gg \ln 2$.

The transition temperature T_C of the rotator model can be estimated for $D = 2$ from the divergence of the polarisability of a vortex pair, which leads to $k_B T_C \sim \pi J_R$, and using equation (13*b*) to

$$\beta^{c}_{DE}J^{i}_{DE} \sim \ln \pi. \tag{43}$$

For $D = 3$ a random walk argument (Banks *et al* 1977, Holz 1978b) for the infinite extension of a vortex loop leads to $k_B T_C \sim \pi J_R(\ln 8 + 2 \times 0{\cdot}5774)/\ln 5$, and using equation (13*b*) to

$$\beta^{c}_{DE}J^{i}_{DE} \sim \ln(2\pi). \tag{44}$$

From the arguments leading to equations (40) and (42), and from equations (43) and (44), it follows that, if the PT of the discrete excitation model occurs in the interval

$$\ln 2 < \beta_{DE}J^{i}_{DE} \leqslant \ln(2D-1), \tag{45}$$

it may have some features in common with the PT occurring in the rotator models.

For $D \geqslant 3$ the PT occurring in the discrete excitation model may be continuous, because the infinite extension of loops may proceed over random walk processes which are hardly modified by 'excluded volume' effects. For $D = 2$ the nature of the PT may deviate more from mean field behaviour owing to the increased importance of 'excluded volume' effects.

Qualitatively this picture may apply to the discrete excitation models constructed for the rotator and XY models. For the Heisenberg model one expects no PT to occur for $D = 2$, as explained by Belavin and Polyakov (1975). A tentative explanation within the discrete excitation model introduced in § 5 may be that the ferromagnetic fluctuations produced by the continuous Ising model are able to prevent an infinite loop extension for $D = 2$. As explained at the end of § 5 a tendency towards such behaviour is also present for $D \geqslant 2$. Accordingly, an additional dimensional argument is necessary to explain the occurrence of a PT for $D \geqslant 3$, and this requires a study of equations (35)–(37). A qualitative argument, however, may be developed by considering an anisotropic Heisenberg model where equation (34*b*) does not apply. Equation (35) still holds in this case, and equation (38) without the factor $\frac{3}{2}$ gives the correlation function $\langle S^x_{0'}S^x_{r'} + S^y_{0'}S^y_{r'}\rangle$. For $\beta_{DE}J^{z}_{DE} \gg \exp(-\beta_{DE}J^{i}_{DE})$ long-range ordering for the correlation function $\langle S^z_{0'}S^z_{r'}\rangle$ may be achieved. From the argument given in § 5 it follows then that the density and growth of loops is curtailed, and no infinite extension of loops may occur. At the other extreme $\beta_{DE}J^{z}_{DE} \ll \exp(-\beta_{DE}J^{i}_{DE})$ an increased density of loops prevents long-range ordering for the correlation function $\langle S^z_{0'}S^z_{r'}\rangle$, but infinite extension of a loop may occur. This makes it plausible that for an intermediate value of the anisotropy, or say the isotropic Heisenberg model, either no PT occurs at all or a PT occurs where infinite loop extension and ferromagnetic ordering appear simultaneously. The first case would apply to $D = 2$ and the second to $D \geqslant 3$, where a symmetry-breaking field would define an ordering direction different from $\theta_{i'} = 0$ or π. If the symmetry-breaking field is taken along $\theta_{i'} = 0$ or π no infinite loop extension will presumably occur.

In the presence of a magnetic field one knows (Dunlop and Newman 1975) that the rotator models do not undergo a PT. A qualitative argument that the same happens for the discrete excitation model follows from the fact that the magnetic field introduces pairs of monopoles of unit strength into the system. A growing loop will now always be unstable against disintegration into open-ended lines decorated by monopoles. This will occur once the loop exceeds a certain critical length which depends on the chemical potential of the monopoles and the gain in entropy by opening a closed loop. One may still expect that the gas of monopoles bound into pairs may undergo a PT into a state of unbounded pairs. Because each unbounded pair would produce an infinitely extended line connecting it, this process would necessarily be stopped by excluded volume effects. This also suggests that for the discrete excitation model in the presence of a magnetic field no PT occurs and line extensions remain finite.

Finally we make a remark with respect to the discrete excitation model introduced in § 2. If the right-hand side of equation (3) extends over all positive and negative integers, then it can be shown (Chui and Weeks 1976) for $D = 2$ that a vortex plasma arises and for $D = 3$ that a vortex loop plasma arises (see e.g. Holz 1978a). In general it can be expected that $(D-2)$-dimensional vortex-like objects arise interacting with a Coulomb interaction $(1/r^{D-2})$. For $D = 2$ and 3 the introduction of the quantities $n_{\langle i,j\ldots\rangle}$ defined in § 2 proved rather useful for the problem and presumably also allows in arbitrary dimensions the derivation of a partition function for the $(D-2)$-dimensional objects. A discussion of the properties of the system in terms of these objects is, however, much more difficult. In particular, in the presence of anisotropy fields additional $(D-1)$-dimensional objects appear representing domain walls. Furthermore, the Coulomb interaction between the $(D-2)$-dimensional vortex-like objects becomes screened. In addition, the $(D-1)$-dimensional surfaces bound by the vortex-like objects assume a surface energy proportional to the anisotropy strength, and develop their own thermodynamic degrees of freedom. It has been shown by Holz (1978b) that a discussion of the PT using the vortex-like objects and its disappearance in the presence of a magnetic field can be done using dimensional arguments similar to those in the present paper. We conclude, however, that the properties of the rotator models are easier to study over the corresponding discrete excitation model. Because the latter is much easier to handle with a finite number of degrees of freedom as given by equation (3), it follows that the model used in this paper is rather attractive.

It should be pointed out that Müller and Helfrich (1978) have developed a high-temperature procedure for the $S = \frac{1}{2}$ Heisenberg model which leads to a formulation in terms of triply counted loops.

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